Derivation of tridiagonal System to be solved in spline Interpolation.

$$S(x) = \begin{cases} S_0(x), & x_0 \le x \le x_1 \\ S_1(x), & x_1 \le x \le x_2 \\ \vdots & \\ S_{n-1}(x), & x_{n-1} \le x \le x_n \end{cases}$$

$$S_j(x) = a_j + b_j(x-x_j) + c_j(x-x_j)^2 + d_j(x-x_j)^3$$

$$x_j \le x \le x_{j+1}$$

$$j = 0, \dots, n-1$$

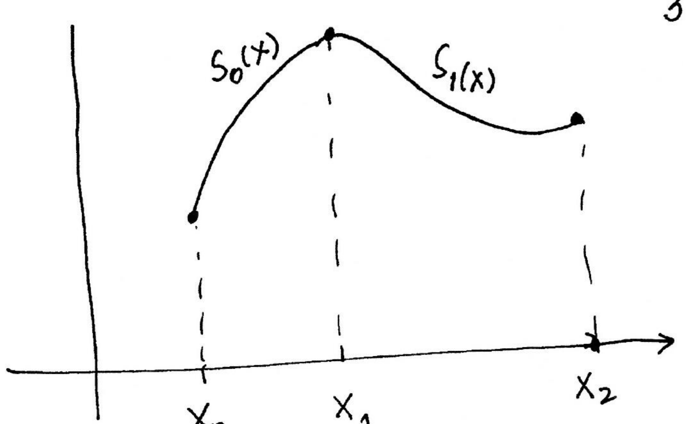
Total of unknowns (a_j, b_j, c_j, d_j) to be determined

$= 4n$

Cont. fn. $S_0(x_1) = S_1(x_1)$
" deriv. $S_0'(x_1) = S_1'(x_1)$
" 2nd deriv. $S_0''(x_1) = S_1''(x_1)$

$n = 2$

8 unkws.



Conditions:

3 interp. conds.

$S(x_j) = f(x_j)$

$j = 0, 1, 2$

Plus two Extra boundary conditions: $\begin{cases} S_0''(x_0) = 0 \;\wedge\; S_1''(x_2) = 0 \\ \text{or} \\ S_0'(x_0) = f'(x_0),\; S_1'(x_2) = f'(x_2) \end{cases}$

In general,

Def.- Given $f(x)$ defined on $[a,b]$ and nodes $a = x_0 < x_1 < \dots < x_n = b$ a Cubic spline interpolant S for f is a function that satisfies

a) $S(x_j) = f(x_j), \quad j = 0, 1, .., n$ $\qquad (n+1)$ conds.

b) $S_{j+1}(x_{j+1}) = S_j(x_{j+1}), \quad j = 0, 1, .., n-2$ $\qquad (n-1)$ conds.

c) $S'_{j+1}(x_{j+1}) = S'_j(x_{j+1}), \quad j = 0, 1, .., n-2$ $\qquad (n-1)$ conds.

d) $S''_{j+1}(x_{j+1}) = S''_j(x_{j+1}), \quad j = 0, 1, .., n-2$ $\qquad (n-1)$ conds.

$\qquad 4n-2$ conds.

Plus BC's.

e) $S''(x_0) = 0, \quad S''(x_n) = 0$ $\qquad$ Free or natural boundary conds.

or

$S'(x_0) = 0, \quad S'(x_n) = 0$ $\qquad$ Clamped boundary cond.

Total # conds $= \underline{4n}$.

Same as # of unknws.

a) Interpolation Condition:

$$S(x_j) = f(x_j), \quad j = 0, 1, .., n$$

implies

$$f(x_j) = a_j + \cancel{b_j (x_j - x_j)}^{0} + \cancel{c_j (x_j - x_j)^2}^{0} + \cancel{d_j (x_j - x_j)^3}^{0}$$

or $\boxed{a_j = f(x_j)}$ $\quad j = 0, 1.., n.$

In particular, $\boxed{a_n = f(x_n)}$

All a_j, $j=0,..,n-1$ are known from the data provided.

b) Continuity of $S(x)$ at intermediate interpolating points

3n unknowns b_j, c_j, d_j $j=0,..,n-1$ to be determined

$$S_{j+1}(x_{j+1}) = S_j(x_{j+1}), \quad j = 0, 1, n-2$$

$$\Rightarrow \quad a_{j+1} = a_j + b_j (x_{j+1} - x_j) + c_j (x_{j+1} - x_j)^2 + d_j (x_{j+1} - x_j)^3.$$

or $\boxed{a_{j+1} = a_j + b_j h_j + c_j h_j^2 + d_j h_j^3}$, $j = 0, 1.., n-2$

Also, using the interpolation condition at $x = x_n$

$$\boxed{a_n = a_{n-1} + b_{n-1} h_n + c_{n-1} h_n^2 + d_{n-1} h_n^3}$$

So, we have

$$\boxed{a_{j+1} = a_j + b_j h_j + c_j h_j^2 + d_j h_j^3}$$

for $j = 0, 1 .., n-1$ $\qquad$ n equs.

C) Continuity of $S'(x)$ at intermediate interpolating points.

$$S'_{j+1}(x_{j+1}) = S'_j(x_{j+1}), \quad j = 0, 1, ., n-2$$

$(n-1)$ equs.

or

$$\boxed{b_{j+1} = b_j + 2c_j h_j + 3d_j h_j^2}$$

we do not have conds. for

$$S'(x_0) = S'_0(x_0) = b_0 = ?$$

and $S'(x_n) = S'_{n-1}(x_n) = b_{n-1} + 2c_{n-1} h_{n-1} + 3d_{n-1} h_{n-1}^2$

$= ?$

by defining

$$\boxed{b_n = S'(x_n),}$$

we obtain

$$\boxed{b_n = b_{n-1} + 2c_{n-1} h_{n-1} + 3d_{n-1} h_{n-1}^2}$$

and

$$\boxed{b_{j+1} = b_j + 2c_j h_j + 3d_j h_j^2,}$$

$j = 0, 1 .., n-1$

n equs.

d) <u>Continuity of $S''(x)$ at intermediate interpolating points</u>

$$S''_{j+1}(x_{j+1}) = S''_j(x_{j+1}), \quad j = 0, 1, .., n-2$$

or $$2c_{j+1} = 2c_j + 6d_jh_j,$$

or $$\boxed{c_{j+1} = c_j + 3d_jh_j}, \quad j = 0, 1, .., n-2$$

$(n-1)$ equs.

We do not have conditions for

$S''(x_0) = ?$ and $S''(x_n) = ?$

$\quad\|\qquad\qquad\quad\|$

$S''_0(x_0) = ?$ $\quad S''_{n-1}(x_n) = ?$

by defining, $\boxed{c_n = S''_{n-1}(x_n)/2}$

$\boxed{c_n = c_{n-1} + 3d_{n-1}h_{n-1}}$

we obtain

and $$\boxed{c_{j+1} = c_j + 3d_jh_j} \quad j = 0, 1, .., n-1$$

n equs.

Summarizing, we have

$b_j, c_j, d_j \quad j = 0, 1 .., n-1$

$3n$ unknowns.

Plus, $\boxed{b_n \text{ and } c_n}$ two more unknowns

and we have

from (b) $\quad n$ equs.

from (c) $\quad n$ equs.

from (d) $\quad n$ equs.

$3n$ equs.

So, we need two additional equs. to have same number of equs. as unknowns.

They are going to be, two boundary conditions:

A) Natural BC's

$$S''(x_0) = S_0''(x_0) = 2c_0 + 6d_0(x_0 - x_0) = f''(x_0)$$

given data

or $\boxed{c_0 = f''(x_0)/2}$

and $S''(x_n) = S_{n-1}''(x_n) = 2c_{n-1} + 6d_{n-1}h_{n-1} = f''(x_n)$

given data

or $2c_n = S''(x_n) = f''(x_n)$

or $\boxed{c_n = f''(x_n)/2}$

(II) Clamped BC's.

$$S_0'(x) = b_0 + 2c_0(x-x_0) + 3d_0(x-x_0)^2$$

$$\Rightarrow \boxed{S_0'(x_0) = b_0 = f'(x_0)}$$

Also $\cancel{\boxed{c_n = \frac{S''(x_n)}{2} = \frac{f''(x_n)}{2}}}$ Clamped B.C.

$$\boxed{b_n = S'(x_n) = f'(x_n)} = S_{n-1}'(x_n)$$

therefore, adding the two bound. conds.

either natural (2) or clamped (2).

we have $3n+2$ conds to obtain

$3n+2$ unknws.

Summarizing

$$a_{j+1} = a_j + b_j h_j + c_j h_j^2 + d_j h_j^3, \qquad j = 0, 1, .., n-1 \quad (1)$$

$$b_{j+1} = b_j + 2c_j h_j + 3d_j h_j^2, \qquad j = 0, .., n-1 \quad (2)$$

$$c_{j+1} = c_j + 3d_j h_j, \qquad j = 0, ., n-1 \quad (3)$$

BC's :

(I) $S''(x_0) = 2c_0 = 0 \Rightarrow \boxed{c_0 = 0}$, $\quad S''(x_n) = 2c_n = 0 \Rightarrow \boxed{c_n = 0}$

or

(II) Clamped Boundary : $b_0 = f'(x_0)$, $b_n = f'(x_n)$.

# of unknws: $\underbrace{2(n+1)}_{b_j\text{'s},\, c_j\text{'s}} + \underbrace{n}_{d_j\text{'s}} = 3n+2$; # of eqns $= 3n+2$

Equs. ① – ③ are combined as follows:

Solving for d_j in ③

$$\boxed{d_j = \frac{1}{3h_j}\left(c_{j+1} - c_j\right),} \qquad j = 0, 1, .., n-1. \quad (*)$$

Subst. in (1) leads to

$$\boxed{a_{j+1} = a_j + b_j h_j + \frac{h_j^2}{3}\left[c_{j+1} + 2c_j\right],} \qquad j = 0, 1, .., n-1. \; (4)$$

Also, substitution of (*) into (2)

$$\boxed{b_{j+1} = b_j + h_j\left(c_{j+1} + c_j\right),} \qquad j = 0, 1, .., n-1 \qquad (5)$$

Solving for b_j in (4)

$$\boxed{b_j = \frac{1}{h_j}\left[a_{j+1} - a_j - \frac{h_j^2}{3}\left(c_{j+1} + 2c_j\right)\right],} \qquad j = 0, 1, .., n-1 \qquad (6)$$

Substitution of (6) into (5). (This is done only for $j = 0, .., n-2$)

$$\boxed{\frac{3}{h_{j+1}}\left(a_{j+2} - a_{j+1}\right) - \frac{3}{h_j}\left(a_{j+1} - a_j\right) = h_j c_j + 2\left(h_j + h_{j+1}\right) c_{j+1} + h_{j+1} c_{j+2}}$$

$$j = 0, 1, .., n-2. \qquad (7)$$

Rewritten (7) decreasing the index by 1.

$$h_{j-1}\, c_{j-1} + 2\,(h_{j-1}+h_j)\, c_j + h_j\, c_{j+1} = \frac{3}{h_j}\,(a_{j+1}-a_j) - \frac{3}{h_{j-1}}\,(a_j - a_{j-1}) \qquad (8)$$

$j=1,\ ,n-1$

$(n-1)$ equs.

n-1 equs.

$$j=1:\quad h_0 c_0 + 2(h_0+h_1)c_1 + h_1 c_2 = \frac{3}{h_1}(a_2-a_1) - \frac{3}{h_0}(a_1-a_0)$$

$$\vdots$$

$$j=n-1\quad h_{n-2}\, c_{n-2} + 2(h_{n-2}+h_{n-1})c_{n-1} + h_{n-1}\, c_n = \frac{3}{h_{n-1}}(a_n-a_{n-1}) - \frac{3}{h_{n-2}}(a_{n-1}-a_{n-2}) \qquad (9)$$

Since $c_0=0$, $c_n=0$ (Nat. boundary conds.). the linear system for the unknowns $c_0,\dots,c_n$ can be written as

$$\begin{bmatrix} 1 & 0 & 0 & 0 & \cdots & \cdots & 0 \\ h_0 & 2(h_0+h_1) & h_1 & 0 & \cdots & \cdots & 0 \\ 0 & h_1 & 2(h_1+h_2) & h_2 & \cdots & \cdots & 0 \\ \vdots & & \ddots & \ddots & \ddots & & \vdots \\ \vdots & & & & & & h_{n-1} \\ \vdots & & & & h_{n-2} & 2(h_{n-2}+h_{n-1}) & h_{n-1} \\ 0 & \cdots & \cdots & \cdots & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ \vdots \\ \vdots \\ c_{n-2} \\ c_{n-1} \\ c_n \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{3}{h_1}(a_2-a_1) - \frac{3}{h_0}(a_1-a_0) \\ \vdots \\ \vdots \\ \frac{3}{h_{n-1}}(a_n-a_{n-1}) - \frac{3}{h_{n-2}}(a_{n-1}-a_{n-2}) \\ 0 \end{bmatrix}$$

$$A\vec{c} = \vec{b} \qquad (10)$$

Since the matrix A of System (10) is strictly diagonally dominant, it is invertible and the System (10) has a unique solution.

For clamped BC's, there is a similar result with changes in the 1^{st} and last equations in the system due to the new BC's.

We have proved the following result:

Thm: ① $f(x)$ defined at $x_0 = a < x_1 < \ldots < x_n = b$. $\longrightarrow$ $S''(a) = 0$, $S''(b) = 0$

then $f(x)$ has a unique natural cubic spline, $S(x)$ which interpolates $f(x)$ on nodes $a = x_0, x_1, \ldots, x_n = b$.